We consider the single-commodity single-origin fixed-charge network flow problem (SSFNP) defined on a directed network $G = (V, A)$, where $V$ is the node set and $A$ is the arc set, with each arc $a$ represented by the ordered pair of head and tail nodes $(h(a), t(a))$. The node set includes a single origin (with no incoming arc), multiple destinations (with no outgoing arc) and transshipment nodes (with both incoming and outgoing arcs). The destinations and the transshipment nodes are grouped in sets $D$ and $T$, respectively. Each destination $k \in D$ has an associated demand $d^k > 0$. On each arc $a \in A$, there is a capacity $u_a$ (with $0 < u_a \leq \sum_{k \in D} d^k$), a per unit transportation cost $c_a \geq 0$ and a fixed cost $f_a \geq 0$ that is incurred whenever some flow is sent through arc $a$. The problem consists in satisfying the demands at minimum cost, while respecting capacity and flow conservation constraints.

By introducing flow variables $x_a$ on the amount of flow on each arc and binary variables $y_a$ that specifies whether arc $a$ is used or not, the SSFNP can be modeled as follows:

\[
\min \sum_{a \in A} (c_a x_a + f_a y_a) \tag{1}
\]

\[
\sum_{a \in B_i} x_a - \sum_{a \in A_i} x_a = \begin{cases} 
  d^i, & i \in D, \\
  0, & i \in T,
\end{cases} \tag{2}
\]

\[
x_a \leq u_a y_a, \quad a \in A, \tag{3}
\]

\[
x_a \geq 0, \quad a \in A, \tag{4}
\]

\[
y_a \in \{0, 1\}, \quad a \in A, \tag{5}
\]

where $A_i = \{a \in A|h(a) = i\}$ and $B_i = \{a \in A|t(a) = i\}$.

This problem has a large number of applications in transportation and logistics, most notably in supply chain design and location-distribution. The classical capacitated fixed-charge network flow problem (CFNP) is defined in a similar way as the SSFN, but might
include multiple origins, each having an associated supply. Any CFNP instance can be cast as an SSFNP instance by adding a super-origin connected to each origin by an arc with zero costs and a capacity equal to the supply of that origin. When the CFNP network is bipartite (i.e., there are no transshipment nodes), we obtain the well-known fixed-charge transportation problem, which is a special case of the CFNP, and therefore also of the SSFNP. Another particular case of the CFNP appears when the supply at each origin and the capacity of each arc are both unlimited; by adding the super-origin, we then obtain an uncapacitated instance of the SSFN, i.e., $u_a = \sum_{k \in D} d^k$ for each arc $a$. The uncapacitated and capacitated facility location problems are two other classical problems that can be represented as SSFNPs by adding a super-origin connected to every potential facility by an arc with no transportation cost, a fixed cost equal to the fixed cost at the facility and a capacity equal to the capacity at the facility.

Branch-and-cut is a popular approach to solve SSFNP and its particular cases: valid inequalities are added to the model and the resulting large-scale (in the number of rows) LP relaxation is solved by a cutting-plane procedure that gradually inserts violated valid inequalities (see, for instance, Ortega and Wolsey (2003)). Another classical approach works through a reformulation of the model with a large number of variables that allows the derivation of valid inequalities developed for the new variable space. The resulting large-scale (in the number of columns, and possibly also in the number of rows) LP relaxation can be solved by a column-and-row generation procedure. This is the approach we take in this paper. Our objective is to tackle large-scale instances with thousands of arcs and hundreds of nodes. As we will see, these dimensions lead to reformulations with more than 100,000 variables and constraints, that can be solved by column-and-row generation embedded in branch-and-bound, thus yielding a branch-and-price-and-cut (B&P&C) algorithm.

We define the variables $w^k_a$, which represent the fraction of the demand $d^k$ for destination $k \in D$ that flows through arc $a$. We then add the following constraints to the above model for SSFNP:

$$w^k_a \in [0, 1], \quad a \in A, k \in D,$$

$$\sum_{k \in D} d^k w^k_a = x_a, \quad a \in A,$$

$$\sum_{a \in B_i} w^k_a - \sum_{a \in A_i} w^k_a = \begin{cases} 1, & k \in D, i = k, \\ 0, & k \in D, i \in T \cup D \setminus \{k\}. \end{cases}$$
The addition of the strong inequalities (9) to the model produces significantly better LP relaxation lower bounds, which is instrumental in solving large-scale instances through enumerative algorithms.

Another approach to obtain the same LP relaxation lower bound consists in characterizing the polyhedron $\text{proj}_{xy}(R_{wxy})$, the projection over the space of $x$ and $y$ variables of the polyhedron $R_{wxy}$ defined by (6)-(10), i.e., the LP relaxation of the uncapacitated SSFNP polyhedron in the space of $w$, $x$ and $y$ variables. The (exponentially many) dicut collection inequalities completely characterize the set $\text{proj}_{xy}(R_{wxy})$ (Rardin and Wolsey, 1993). Thus, in case efficient separation algorithms are available for these inequalities, one might solve by a cutting-plane algorithm the LP relaxation obtained by adding the dicut collection inequalities to model (1)-(4) and (10). To see why this LP relaxation lower bound is equal to that obtained by the reformulation, it suffices to remark that $R_{wxy} \cap (3) \equiv \{6\} - \{10\} \cap (3) = \{3\}, \{6\} - \{7\}, \{9\} - \{10\} \cap (8) \equiv \text{conv}(Q_{wxy}) \cap (8)$.

After projecting out the $x$ variables, the reformulation can be simplified to:

$$\min \sum_{a \in A} \sum_{k \in D} c_a w_k^a + \sum_{a \in A} f_a y_a \tag{11}$$

subject to constraints (5)-(6), (8)-(9) and

$$\sum_{k \in D} d^k w_a^k \leq u_a y_a, \quad a \in A. \tag{12}$$

This multicommodity flow reformulation belongs to the folklore (Rardin and Choe, 1979), but is also notoriously difficult to solve. Based on the recent work of Gendron and Larose (2012), we propose a B&P&C algorithm to solve this large-scale formulation. Conceptually, the restricted master problem solved at each column generation iteration is obtained directly from the compact arc-based model by considering only a subset of the commodity flow variables. The pricing subproblem corresponds to a Lagrangian relaxation of the flow conservation equations, (8), and the capacity constraints (12), leaving in the Lagrangian subproblem only the strong inequalities. The column generation procedure is completed by a cut generation step based on strong inequalities. The computational results on randomly generated multicommodity instances show that this approach is competitive with branch-and-cut methods and outperforms them when the number of commodities is large, say more than 100.

We will present computational results on a large set of single-commodity randomly generated instances, as well as on instances derived from various applications in supply chain design and location-distribution. The latter will include multiple-origin capacitated and uncapacitated instances, fixed-charge transportation problems, and capacitated and uncapacitated facility location instances.

Although the multicommodity reformulation of SSFNP is classical, it has been considered “intractable” for large-scale instances and branch-and-cut algorithms on the original
variable space were used instead. The proposed B&P&C algorithm can efficiently solve instances with thousands of arcs and hundreds of nodes, and possibly larger instances. This perspective opens a number of interesting avenues, since the multicommodity reformulation of single-commodity fixed-charge problems is often overlooked or dismissed as a modeling alternative, because of its reputation. The suggested B&P&C algorithm, or adaptations of it, can be used to solve a wide variety of problems in transportation and logistics that can be represented as large-scale single-commodity fixed-charge network flow instances. Also noteworthy in this work is the fact that the multicommodity flow model is not solved as a Dantzig-Wolfe reformulation (thus leading to the classical path-based model), but rather addressed directly, i.e., the compact arc-based model is the master problem of the column generation procedure; this approach can be seen as a particular case of two recently proposed column-and-row generation frameworks, the structured Dantzig-Wolfe method of Frangioni and Gendron (2012) and the algorithm for problems with column-dependent-rows (Muter et al., 2012).

References


